COMBINATIONAL COMBINATIONS.

BY

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PREFACE.

Algebra treats, in most parts, how to combine the symbols of quantities; and the theory of combination especially deals with certain laws of combining several things or numbers.

In this pamphlet, I shall explain some principles relating to complicate cases of combinations premising certain truth as acquainted.

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Tokyo, January 27th, 1900.

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COMBINATIONAL THEOREM.

Additional Corrigenda.

Read,	for,	page,	line.
${}^nC_r$,	C_r,	3.	4.
$+{}^{n+r-3}C_{r-1}+{}^{n+r-2}C_{r-1}$,	$+{}^{n-2}C_{r-1}+{}^{n-1}C_{r-1}$,	7.	21.

of x and y, it must be true for all values of the same variables.

2. Substitute $-x$ and $-y$ for x and y in the formula (A), conceive to change the order of consecutive factors in each of permutational factors, and divide the both sides of the equality by $(-1)^{n+1}$; then

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CHAPTER I.

COMBINATIONAL THEOREM.

1. Expand the identity

$$(1-z)^{-(x+y)} \equiv (1-z)^{-x}(1-z)^{-y}$$

and equate the coefficients of z^{n+1} in the both sides; then we have

$$\begin{aligned} {}^{x+y}H_{n+1} = {}^{x}H_{n+1} + {}^{x}H_{n}\,{}^{y}H_{1} + {}^{x}H_{n-1}\,{}^{y}H_{2} + \cdots\cdots \\ + {}^{x}H_{2}\,{}^{y}H_{n-1} + {}^{x}H_{1}\,{}^{y}H_{n} + {}^{y}H_{n+1}. \end{aligned}$$

Multiply each side of this equality by $\underline{|n+1}$; then $\left(\text{since } {}^{x}H_r = \dfrac{(x+r-1)_r}{\underline{|r}},\right)$

$$\begin{aligned} (x+y+n)_{n+1} = (x+n)_{n+1} + {}^{n+1}C_1(x+n-1)_n y_1 \\ + {}^{n+1}C_2(x+n-2)_{n-1}(y+1)_2 + \cdots\cdots \\ + {}^{n+1}C_{n-1}(x+1)_2(y+n-2)_{n-1} + {}^{n+1}C_n x_1 (y+n-1)_n \\ + (y+n)_{n+1} \qquad\qquad (A). \end{aligned}$$

Since the formula (A) is true for more than $n+1$ values of x and y, it must be true for all values of the same variables.

2. Substitute $-x$ and $-y$ for x and y in the formula (A), conceive to change the order of consecutive factors in each of permutational factors, and divide the both sides of the equality by $(-1)^{n+1}$; then

$$(x+y)_{n+1}=x_{n+1}+{}^{n+1}C_1x_ny_1+{}^{n+1}C_2x_{n-1}y_2+\cdots\cdots\cdots$$
$$+{}^{n+1}C_{n-1}x_2y_{n-1}+{}^{n+1}C_nx_1y_n+y_{n+1}.$$

Hence the formula (A) is the converse of Vandermonde's Theorem; so that each proves the truth of other for negative values of x and y.

3. The whole number of ways of selecting x things r at a time is known to be ${}^xC_r$.

Suppose x things are divided into n groups, there being y, z, w, etc. things respectively; then r things can be taken, in all possible ways, out of one or more of the n groups.

Now, the number of taking r things out of each group separately is clearly $\Sigma\,{}^yC_r$ altogether.

Also, the number of taking r things out of every two groups is $\Sigma\,{}^yC_{r-s}\,{}^zC_r$ for any particular integral values of r and s; therefore, if we adopt a notation $N[\Sigma^yC_{r-s}\,{}^zC_s]$ for the sum

$$\Sigma\,{}^yC_{r-1}\,{}^zC_1+\Sigma^yC_{r-2}\,{}^zC_2+\cdots\cdots\cdots+\Sigma^yC_{r-s}\,{}^zC_s+\cdots\cdots\cdots,$$

provided $s \ngtr r-s$, the total number of taking r things out of every two groups may be denoted by the sum $N[\Sigma^yC_{r-s}\,{}^zC_s]$.

Similarly, since the number of taking out of every three groups is $\Sigma^yC_{r-s}\,{}^zC_{s-t}\,{}^wC_t$ for any particular integral values of r, s and t; the total number for all possible values of r, s and t, provided $t \ngtr s-t \ngtr r-s$, may be denoted by $N[\Sigma^yC_{r-s}\,{}^zC_{s-t}\,{}^wC_t]$.

And, in general, the total number of taking r things out of every k groups may be denoted by

$$N[\Sigma({}^yC_{r-s}\,{}^zC_{s-t}\,{}^wC_{t-p}\cdots\cdots)],\ \text{provided}$$
$$r-s \geqq s-t \geqq t-p \geqq \cdots\cdots\cdots,$$

each term of the sum consisting of k combinational factors.

Hence, we have the following

Formula.

$$C_r \equiv \Sigma^y C_r + N[\Sigma^y C_{r-s}{}^z C_s] + N[\Sigma^y C_{r-s}{}^z C_{s-t}{}^w C_t]$$
$$+ \cdots\cdots\cdots + N[\Sigma({}^y C_{r-s}{}^z C_{s-t}{}^w C_{t-p} \cdots\cdots\cdots)] +$$
$$\cdots\cdots\cdots \qquad (B),$$

where $x = y + z + w + \cdots\cdots$, and

$$r - s \overline{\overline{>}} s - t \overline{\overline{>}} t - p \overline{\overline{>}} \cdots\cdots\cdots.$$

Multiply each side of the formula (B) by $\underline{|r}$; then

$$(y + z + w + \cdots\cdots)_r \equiv$$
$$\Sigma y_r + N[{}^r C_s \Sigma y_{r-s} z_s] + N[{}^r C_s {}^s C_t \Sigma y_{r-s} z_{s-t} w_t]$$
$$+ \cdots\cdots + N[({}^r C_s {}^s C_t {}^t C_p \cdots\cdots) \Sigma(y_{r-s} z_{s-t} w_{t-p} \cdots\cdots)]$$
$$+ \cdots\cdots\cdots \qquad (B').$$

This is true for all values of y, z, w, etc.

Example. When $n=2$, the formula (B') becomes

$$(y+z)_r \equiv \Sigma y_r + N[{}^r C_s \Sigma y_{r-s} z_s]$$

This is the contracted form of Vandermonde's Theorem.

4. By means of the formula (B'), the Binomial Theorem

$$(1+x)^m (1+x)^n (1+x)^l \cdots\cdots = (1+x)^{m+n+l+\cdots\cdots}$$

can be directly proved for all values of m, n, l, etc.

Conversely, we can prove the truth of (B′) as follows.

Consider the product

$$(1-x)^{-y}(1-x)^{-z}(1-x)^{-w} \cdots\cdots =$$
$$(1 + {}^y H_1 x + {}^y H_2 x^2 + \cdots\cdots + {}^y H_r x^r + \cdots\cdots\cdots) \times$$
$$(1 + {}^z H_1 x + {}^z H_2 x^2 + \cdots\cdots + {}^z H_r x^r + \cdots\cdots\cdots) \times$$
$$(1 + {}^w H_1 x + {}^w H_2 x^2 + \cdots\cdots + {}^w H_r x^r + \cdots\cdots\cdots) \times$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots;$$

the coefficients of x^{r+1} in both sides can be seen to make up the equality

$$
\begin{aligned}
&{}^{(y+z+w+\ldots\ldots)}H_{r+1}= \\
&\Sigma^{y}H_{r+1}+N[\Sigma^{y}H_{r-s+1}\,{}^{z}H_{s}]+N[\Sigma^{y}H_{r-s+1}{}^{z}H_{s-t}{}^{w}H_{t}] \\
&+\cdots\cdots\cdots\cdots+N[\Sigma({}^{y}H_{r-s+1}\,{}^{z}H_{s-t}{}^{w}H_{t-p}\ldots\ldots)]+ \\
&\cdots\cdots\cdots\cdots\qquad\qquad (C),
\end{aligned}
$$

provided $r-s+1\gtreqless s-t\gtreqless t-p\gtreqless\cdots\cdots\cdots$.

Multiply each side of (C) by $\underline{|r+1}$; then

$$
\begin{aligned}
&(y+z+w+\cdots\cdots+r)_{r+1}= \\
&\Sigma(y+r)_{r+1}+N[{}^{r+1}C_{s}\Sigma(y+r-s)_{r-s+1}(z+s-1)_{s}] \\
&+N[{}^{r+1}C_{s}\,{}^{s}C_{t}\Sigma(y+r-s)_{r-s+1}(z+s-t-1)_{s-t}(w+t-1)_{t}] \\
&+\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots+ \\
&N[({}^{r+1}C_{s}\,{}^{s}C_{t}\,{}^{t}C_{p}\cdots\cdots)\Sigma\{(y+r-s)_{r-s+1}(z+s-t-1)_{s-t} \\
&(w+t-p-1)_{t-p}\cdots\cdots\cdots\}]+\cdots\cdots\qquad (C')
\end{aligned}
$$

In (C′), replace y, z, w, etc. by $-y$, $-z$, $-w$, etc., and conceive to change the order of consecutive factors in each of permutational factors ; then

$$
\begin{aligned}
&(-1)^{r+1}(y+z+w+\cdots\cdots)_{r+1}= \\
&\quad(-1)^{r+1}\{\Sigma y_{r+1}+N[{}^{r+1}C_{s}\Sigma y_{r-s+1}z_{s}]+\cdots\cdots \\
&\qquad+N[{}^{r+1}C_{s}{}^{s}C_{t}\,{}^{t}C_{p}\ldots\ldots y_{r-s+1}\,z_{s-t}w_{t-p}\ldots\ldots] \\
&\qquad+\cdots\cdots\cdots\cdots\}
\end{aligned}
$$

This proves the truth of the combinational theorems [(B') and (C')] for opposite signs ; because, in $\Pi(1-x)^{-y}$ and $\Pi(1-x)^{y}$, the coefficients of x^{r+1} are clearly ${}^{(y+z+w+\ldots\ldots)}H_{r+1}$ and $(-1)^{r+1}\,{}^{(y+z+w+\ldots\ldots)}C_{r+1}$.

CHAPTER II.

FIGURATE NUMBERS IN DETERMINANT.

5. The series of figurate numbers

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & . & . & . & . \\
1 & 2 & 3 & 4 & 5 & . & . & . & n \\
1 & 3 & 6 & 10 & 15 & . & . & . & . \\
1 & 4 & 10 & 20 & 35 & . & . & . & . \\
1 & 5 & 15 & 35 & 70 & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & n & . & . & . & . & . & . & {}^{n}H_{n-1},
\end{array}$$

since

$$1+{}^{n-1}H_1+{}^{n-1}H_2+\cdots\cdots+{}^{n-1}H_{r-1}+{}^{n-1}H_r={}^{n}H_r,$$

may be written as

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & . & . & . & 1 \\
1 & {}^{2}H_1 & {}^{2}H_2 & {}^{2}H_3 & {}^{2}H_4 & . & . & . & {}^{2}H_{n-1} \\
1 & {}^{3}H_1 & {}^{3}H_2 & {}^{3}H_3 & {}^{3}H_4 & . & . & . & {}^{3}H_{n-1} \\
1 & {}^{4}H_1 & {}^{4}H_2 & {}^{4}H_3 & {}^{4}H_4 & . & . & . & {}^{4}H_{n-1} \\
1 & {}^{5}H_1 & {}^{5}H_2 & {}^{5}H_3 & {}^{5}H_4 & . & . & . & {}^{5}H_{n-1} \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
1 & {}^{n}H_1 & {}^{n}H_2 & {}^{n}H_3 & {}^{n}H_4 & . & . & . & {}^{n}H_{n-1}
\end{array} \quad (D).$$

Also, since

$$^nH_r = 1 + {}^2H_{r-1} + {}^3H_{r-1} + \cdots\cdots + {}^{n-1}H_{r-1} + {}^nH_{r-1},$$

the table (D) may be reduced to

$$\begin{array}{cccccccccc}
1 & 1 & 1 & 1 & 1 & . & . & . & 1 & \\
1 & {}^2H_1 & {}^3H_1 & {}^4H_1 & {}^5H_1 & . & . & . & {}^nH_1 & \\
1 & {}^2H_2 & {}^3H_2 & {}^4H_2 & {}^5H_2 & . & . & . & {}^nH_2 & \\
1 & {}^2H_3 & {}^3H_3 & {}^4H_3 & {}^5H_3 & . & . & . & {}^nH_3 & \\
1 & {}^2H_4 & {}^3H_4 & {}^4H_4 & {}^5H_4 & . & . & . & {}^nH_4 & \\
. & . & . & . & . & . & . & . & . & \\
. & . & . & . & . & . & . & . & . & \\
. & . & . & . & . & . & . & . & . & \\
1 & {}^2H_{n-1} & {}^3H_{n-1} & {}^4H_{n-1} & {}^5H_{n-1} & . & . & . & {}^nH_{n-1} & (D').
\end{array}$$

This shows that, in the table of figurate numbers, the rth order corresponds to the rth column.

6. Consider the series of expressions

$$\begin{gathered}
1,\\
1 + x,\\
1 + {}^2H_1x + x^2,\\
1 + {}^3H_1x + {}^3H_1x^2 + x^3,\\
1 + {}^4H_1x + {}^3H_2x^2 + {}^4H_1x^3 + x^4,\\
. \quad . \quad . \quad . \quad . \quad .\\
1 + {}^rH_1x + {}^{r-1}H_2x^2 + \cdots\cdots + {}^{r-1}H_2x^{r-2} + {}^rH_1x^{r-1} + x^r;
\end{gathered}$$

since ${}^{r-s}H_{s+1} = {}^rC_{s+1}$, they will be found to be equivalent to

$$(1+x)^0,$$
$$(1+x)^1,$$
$$(1+x)^2,$$
$$(1+x)^3,$$
$$\cdots\cdots\cdots,$$
$$(1+x)^r.$$

Hence, the table (D) is equivalent to

$$\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & . & . & . & 1 \\
1 & {}^2C_1 & {}^3C_2 & {}^4C_3 & . & . & . & {}^rC_{r-1} & . \\
1 & {}^3C_1 & {}^4C_2 & . & . & . & {}^rC_{r-2} & . & . \\
1 & {}^4C_1 & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . \\
. & . & {}^rC_2 & . & . & . & . & . & . \\
. & {}^rC_1 & . & . & . & . & . & . & . \\
1 & . & . & . & . & . & . & . & {}^{2r}C_r
\end{array} \quad (D'').$$

7. The following relations are plain from the tables (D), (D') and (D''):

(1) ${}^nH_r = 1 + {}^rH_1 + {}^rH_2 + \cdots\cdots + {}^rH_{n-2} + {}^rH_{n-1}$,

(2) ${}^nH_r = 1 + {}^{n-1}C_1 + {}^nC_2 + \cdots\cdots + {}^{n+r-3}C_{r-1} + {}^{n+r-2}C_r$,

(3) ${}^nH_r = 1 + {}^rC_{r-1} + {}^{r+1}C_{r-1} + \cdots\cdots + {}^{n-2}C_{r-1} + {}^{n-1}C_{r-1}$.

8. The figurate numbers in determinant.

The determinant in which the successive rows are the series of figurate numbers as

$$\begin{vmatrix} 1 & 1 & 1 & 1 & \cdot & \cdot & \cdot \\ 1 & {}^2H_1 & {}^2H_2 & {}^2H_3 & \cdot & \cdot & \cdot \\ 1 & {}^3H_1 & {}^3H_2 & {}^3H_3 & \cdot & \cdot & \cdot \\ 1 & {}^4H_1 & {}^4H_2 & {}^4H_3 & \cdot & \cdot & \cdot \\ \cdot & \cdot & & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix} = |F\infty|,$$

represents the elimination of the successive integral powers of $\frac{1}{1-x}$ to infinity.

To prove $|F\infty| = 1$.

Assume

$$\frac{1}{1-x} = X = 0,$$

$$\left(\frac{1}{1-x}\right)^2 = X^2 = 0,$$

$$\left(\frac{1}{1-x}\right)^3 = X^3 = 0,$$

etc.,

and introduce a complemental X^0 for convenience ; then the elimination of X^0, X, X^2, X^3, etc. may be denoted by

$$\begin{vmatrix} 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & 1 \end{vmatrix} = 1.$$

If the first row and column for $X^0=1$ be supressed, still the determinant will reserve the similar form.

Hence

$$|F\infty| = |1| = 1.$$

Theorem. *The value of figurate numbers in determinant is always equivalent to unity.*

To prove $|F_4| = 1$.

By modifying the notation in (D'')

$$|F_4| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 1 & {}^2C_1 & {}^3C_1 & {}^4C_1 \\ 1 & {}^3C_1 & {}^4C_2 & {}^5C_2 \\ 1 & {}^4C_1 & {}^5C_2 & {}^6C_3 \end{vmatrix};$$

by subtracting all the elements of each row from the corresponding elements of next row (since ${}^nC_r - {}^{n-1}C_r = {}^{n-1}C_{r-1}$,)

$$|F_4| = \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & {}^2C_1 & {}^3C_1 \\ 0 & 1 & {}^3C_2 & {}^4C_2 \\ 0 & 1 & {}^4C_1 & {}^5C_3 \end{vmatrix} = \begin{vmatrix} 1 & {}^2C_1 & {}^3C_1 \\ 1 & {}^3C_2 & {}^4C_2 \\ 1 & {}^4C_1 & {}^5C_3 \end{vmatrix};$$

by subtracting all the elements of each column from the corresponding elements of next column,

$$\begin{vmatrix} 1 & {}^2C_1 & {}^3C_1 \\ 1 & {}^3C_2 & {}^4C_2 \\ 1 & {}^4C_1 & {}^5C_3 \end{vmatrix} =$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & {}^2C_1 & {}^3C_1 \\ 1 & {}^3C_1 & {}^4C_2 \end{vmatrix} \text{(since } {}^3C_2 = {}^3C_1 \text{ and } {}^4C_1 = {}^4C_3\text{)}$$

$$= |F_3|.$$

Hence

$$|F_4| = |F_3|.$$

Similarly

$$|F_3| = |F_2|;$$

and

$$|F_2| = \begin{vmatrix} 1 & 1 \\ 1 & {}^2C_1 \end{vmatrix} = 1.$$

Therefore

$$|F_4| = 1.$$

This proof is perfectly general.